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Discounting expected values with parameter uncertainty

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Abstract

In valuing future cash flows, the standard practice is to take the current cash flow and then extrapolate at an expected growth rate, which can vary at different points in time. This practice stems from the standard way of dealing with time value of money problems under certainty. However, with uncertain cash flows, this practice underestimates the expected cash flows when the growth rates are serially correlated. As a result, both value and the equity cost, calculated as an internal rate of return, are biased low. Given the prevalence of serial correlation in the economy, this paper demonstrates how to incorporate the effects of serial correlation in a simple way and demonstrates by way of a simulation that the effects can be significant. As a result, it casts doubt on the usefulness of several standard valuation approaches and results.

Keywords: Cash flow; Growth rate; Equity cost

1. Discounting expected values with parameter uncertainty

Discounting (expected future) cash flows (DCF) is at the core of modern finance both for valuation purposes and for estimating internal rates of return for use as discount rates. For example, the constant growth version of the DCF model is commonly used for valuing low risk firms and for calculating their cost of equity capital. As Myers and Borucki (1994) point out, "The discounted cash flow (DCF) formula is the most widely used

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approach to estimate the cost of equity capital to regulated firms in the United States". It is discussed in every introductory finance text and is at the heart of many valuation models, including quite sophisticated corporate models. However, it is well known that although correct for valuing risk-free securities, like government bonds, the DCF model does not handle uncertainty very well. The reason for this is that, in contrast to risk based models, the effects of uncertainty are implicit, either through their impact on market values or the required rate of return. However, even this implicit incorporation of uncertainty fails to recognise parameter uncertainty, that is, how uncertainty affects the estimation of the model's key parameters.

Incorporating parameter uncertainty in a realistic way into the expected cash flow stream is the main objective of this paper. The motivation for doing this is quite simple: without taking parameter uncertainty into account conventional applications of the DCF model consistently underestimate the expected future cash flow stream and as a result underestimate both value and the discount rate when calculated as an internal rate of return. This underestimation of the DCF equity cost can be ball-parked by thinking in terms of the overall market, since there are constraints on the market that do not exist for individual stocks. For example, the long-run real growth rate in the economy is normally pegged at about 4.0%; with 1.0–2.0% inflation, this means a nominal long-run profit growth of about 5.0–6.0%, otherwise corporate profits would indefinitely increase each year as a share of GDP. With an S&P500 average dividend yield under 1.0%, this would put the nominal DCF equity cost for the "market" at less than 7.0%, which is only marginally more than the treasury bond yield. The fact that experienced equity returns indicate a considerably higher realized risk premium indicates that there may be a problem with using this simple application of the DCF model. This could also be why in the current low inflationary environment, the DCF model has to some extent fallen out of favor with regulatory boards. Myers and Borucki (1994), for example, point out how the DCF estimates they obtain are all well below the actual returns on equity earned by regulated firms to the extent that they wonder whether regulators just pay "lip service" to the accepted theory of regulation.

The main reason why parameter uncertainty affects the DCF model is that there is pervasive serial correlation in the economy. As the economy varies through time, profits and dividends reflect the serial correlation of the business cycle. As a result, the expected cash flow stream is greater than that estimated by simple extrapolation. Incorporating parameter uncertainty in the presence of serially correlated growth rates produces a relatively simple and intuitive generalization of the constant growth DCF model.

The organization of the paper is as follows. Section 2 discusses the general valuation approach and the central problem in estimating expected cash flows. Section 3 derives a simple formula to incorporate growth rate uncertainty assuming that both the growth rate and the cash flows each period are normally distributed. Section 4 provides estimates from a simple simulation to develop a deeper understanding of the model and examines the robustness of the approximation used in Section 3. Section 5 looks at some empirical results and discusses how these results affect valuation problems and Section 6 adds some conclusions and suggestions for further research.

2. Valuation approaches

Most valuation problems involve multiple time periods and the standard approach in finance is to specify the problem as,

$$V = \sum_{t=1}^{T} \frac{E[C_t]}{(1+K)^t} \tag{1}$$

where value is determined by discounting the period t expected cash flows $E(C_t)$ over a specified time horizon, T, which can extend to infinity, at a risk adjusted discount rate, K. Extensive research has examined how the discount rate is determined in the multiperiod problem. For example, both Mossin (1968) and Fama (1970) have shown how different types of multiperiod valuation problems can be collapsed into a properly specified single period problem, whose solution involves discounting an expected value with a risk adjusted discount rate. Subsequent work, for example, Myers and Turnbull (1977), has focussed on the assumptions required to use the CAPM required rate of return as the discount rate for valuing multiperiod cash flows. However, the basic approach of discounting an expected cash flow with an appropriate discount rate is preserved. What is important is that it is the expected value, not the modal or median value, that is being discounted. This is the focus of this paper: not how the discount rate is determined, but how the expected cash flows are estimated.

The basic structure of the valuation problem requires that we estimate each period's expected cash flow, that is, we need the expected cash flow in each of period 1, period 2, etc. Hence, we need to understand the process that determines the cash flows each period. In practice, it is extremely difficult to forecast cash flow independently for each year, so that a variety of simplifications are used. Gordon (1962), for example, introduced the simplification that there is a long-run expected growth rate, g, for the cash flows. The formula for a geometric series then allows Eq. (1) to collapse to the familiar constant growth model,

$$V = \frac{E[C_1]}{K-g} \tag{2}$$

where the expected cash flow for time period 1 is valued using a growth adjusted discount rate. Alternative assumptions about the growth rate give rise to the multistage and finite growth rate models.

There are a large number of plausible assumptions for the expected growth rate, each of which give rise to a variation on the basic discounted cash flow model of Eq. (1). However, they all have in common the basic assumption that the growth rate, although different for different periods, is still certain, for example, that the cash flows grow at 10% for 5 years and then 5% forever. None of the extant models take into account the uncertainty in the expected growth rate itself, that is, that the growth rate may have an expected value of 10% or 5%, but is itself stochastic.

In this research, we show that in general, the expected cash flow stream will be higher than the current cash flow extrapolated forward at the expected growth rate. That is, that the conventional "certainty" approach, used in textbooks, casebooks and practice, of extrapolation based on expected growth rates underestimates the expected cash flow stream and as a result underestimates the value of an investment. Conversely, when the

DCF equation is reversed to calculate the discount rate, as an internal rate of return that sets the present value of the expected cash flow stream equal to the market price, the discount rate will also be underestimated.

The central intuition of the paper can be developed by analyzing the first few expected cash flows. For example, taking the first period's expected cash flow, it is simply the current cash flow times the growth expected for time period 1,

$$E[C_1] = E[C_0(1+g_1)] \tag{3}$$

where g_1 is the uncertain growth rate for time period one. Simplifying this equation is straightforward, since there is only one source of uncertainty. As a result, the expectation operator can be taken through and the expected cash flow is simply the current cash flow extrapolated at the expected growth rate for time period 1.

For the second period, the expected cash flow is

$$E[C_2] = E[C_0(1+g_1)(1+g_2)] \tag{4}$$

In form, this is the same as before. However, there are now two sources of uncertainty, that for the growth rate in period 1, as well as that for time period 2. Expanding the expectation, we get

$$E[C_2] = C_0(1+E[g_1])(1+E[g_2]) + C_0\mathrm{Cov}(g_1,g_2) \tag{5}$$

where $\mathrm{Cov}(g_1,g_2)$ is the covariance between the growth rates in time periods 1 and 2. Only if the growth rates are serially uncorrelated will the covariance be zero and the expected cash flow equal to the current cash flow compounded forward at the two expected growth rates.

Eq. (5) shows the central problem in forecasting expected cash flows: if there is serial correlation in the growth rates the expected cash flow stream will generally be higher than that obtained from simply extrapolation at the expected growth rate. This comment is general to any valuation problem that requires discounting expected cash flows and applies to the constant, multistage, and finite growth models, as well as any model that uses an expected growth rate. However, our focus will be on the constant growth model. This is for two reasons: first it is one of the most important models in its own right, particularly as an equity cost model, and second even in finite or multistage growth models, the terminal value is almost always set using the constant growth model. As a result, it is an important component of more general models.

The obvious counterpart to the constant growth model is the constant expected growth model, that is, that $E[g_t] = g$, where the unsubscripted growth rate is the constant expected growth rate. In this case, Eq. (5) becomes

$$E[C_2] = C_0(1+g)^2 + C_0\mathrm{Cov}(g_1,g_2) \tag{6}$$

where the covariance term indicates the bias involved in simple extrapolation.

How we simplify this covariance depends on the assumptions imposed on the uncertainty in the growth rate. A variety of formulations are possible, but for our purposes, we assume the simplest possible process that captures the basic intuition:

$$g_t = (1-\rho)g + \rho g_{t-1} + \epsilon_t \tag{7}$$

where ρ is the serial correlation coefficient and ϵ the random error term. By assumption, the random error term is assumed to be normally distributed, iid, with a mean of zero. Eq. (7) is a simple autoregressive model with one minus the serial correlation coefficient measuring the tendency of the growth rate to return to its "long-run" expected level. It has two interesting extremes, where the serial correlation is either zero or one. If there is no serial correlation, $\rho=0$, the covariance in Eq. (6) is zero and the simple constant growth model is unbiased. On the other hand, perfect serial correlation, $\rho=1$, means that the growth rate is a simple random walk. Eq. (7), therefore, includes the "standard" DCF model, as well as a more general one in which the model is biased due to serial correlation.

If this stochastic growth assumption is substituted into the covariance term in Eq. (6) and we start at the long-run average growth rate (g), we get,

$$E[C_2] = C_0(1+g)^2 + C_0\rho\sigma^2 \tag{8}$$

where σ^2 is the variance of the random error term in Eq. (7) and ρ the serial correlation coefficient. If the growth rates have positive serial correlation, the cash flow in the second period will be unambiguously greater than if they are uncorrelated. Moreover, the effect of the serial correlation is compounded by the uncertainty in the error term. As a result, the bias created by ignoring the uncertainty in the expected growth rate will differ systematically across firms due to differences in both the serial correlation coefficient, as well as the uncertainty in their growth rates.

Eq. (8) is perfectly general. Although the specific equation stems from the correlation structure assumed in Eq. (7), different types of dependencies across time will just produce a different structure of covariance terms; they will not disappear.

3. A generalized constant expected growth model

The expected cash flow in time period 3 is simply,

$$E[C_3] = E[C_2](1+g) + \mathrm{Cov}(C_2, g_3) \tag{9}$$

where again the expected cash flow is just the previous expected cash flow compounded forward at the expected growth rate plus a covariance term. The covariance term will always consist of the covariance between the prior cash flow and the contemporaneous growth rate. For the third period cash flow, the uncertainty stems from the uncertain growth rates in both time periods 1 and 2. This covariance can be simplified due to a result from Stevens (1971) for multiplicative normally distributed random variables[1]

$$\mathrm{Cov}(C_2, g_3) = E[C_1]\mathrm{Cov}(g_2, g_3) + C_0(1+g)\mathrm{Cov}(g_1, g_3) \tag{10}$$

[1] Steven's result states $\mathrm{Cov}(ab,c) = E[a]\mathrm{Cov}(b,c) + E[b]\mathrm{Cov}(a,c)$.

The first covariance is the same as in Eq. (8) and represents the immediate impact of the serial correlation between the growth rates in time periods 2 and 3, this is just $E[C_1]\rho\sigma^2$. The second covariance picks up the impact of the growth rate in time period 1 on that in time period 3. By repeated substitution, this is $C_0(1+g)\rho^2\sigma^2$. By combining these results, we get,

$$E[C_3] = E[C_2](1+g) + E[C_1]\rho\sigma^2 + C_0(1+g)\rho^2\sigma^2 \tag{11}$$

This result is quite intuitive. The third period expected cash flow is the prior expected cash flow compounded forward for one period plus two additional terms reflecting the serial correlation. Suppose, for example, the first period's growth rate is randomly high, since the growth rates are serially correlated this will affect the growth rates in both time periods 2 and 3 through the serial correlation coefficient. If the second period's growth rate is again randomly high, then the third period's cash flow will be affected by the higher growth rates in both prior periods. Since each expected cash flow is determined by the expectation of the prior cash flow and the growth rate, this intertemporal dependence increases the expected cash flows. However, note that the covariance terms will not be the same for each period, since the serial dependence of the growth rates will first build as a result of the uncertainty in the first few growth rates and then stabilize as the memory in the growth rates decays. Of course, how these terms change depends on the size of the serial correlation coefficient and the type of dependence assumed.

To simplify the covariance, note that the expected third period cash flow is affected by both the prior expected cash flows and the serial correlation in the growth rate. In fact, both terms contain the serial correlation coefficient, and the variance in the error term and can be viewed as "coefficients" on both the prior expected cash flows, in this case for those in time periods 1 and 2. For an arbitrary time period T, the expanded expectation of the cash flow is

$$E[C_T] = E[C_{T-1}](1+g) + \mathrm{Cov}(C_{T-1}, g_T) \tag{12}$$

where C_{T-1} can be viewed as the product of two random variables $C_{T-2}(1+g_{T-1})$ If both these variables are assumed to be normally distributed, then Steven's simplification holds and a similar convolution of terms results.[2] In each case, the first term is the prior expected cash flow times one plus the expected growth rate, the second term is the expected cash flow two periods earlier times the serial correlation coefficient and the variance term, while the third term is the cash flow three periods earlier times the serial correlation coefficient squared, the variance in the error term and one plus the growth rate, and the fourth term is the expected cash flow four periods earlier times the serial correlation coefficient cubed, the variance in the error term and one plus the growth rate squared.

[2] This is approximate, since C_{T-1} is the product of $T-2$ normally distributed variables, which converges to a lognormal.

The following table shows how the expected cash flow builds over time.

	C_0	$E[C_1]$	$E[C_2]$	$E[C_3]$	$E[C_4]$
$E[C_1]$	$1+g$				
$E[C_2]$	$\rho\sigma^2$	$1+g$			
$E[C_3]$	$(1+g)\rho^2\sigma^2$	$\rho\sigma^2$	$1+g$		
$E[C_4]$	$(1+g)^2\rho^3\sigma^2$	$(1+g)\rho^2\sigma^2$	$\rho\sigma^2$	$1+g$	
$E[C_5]$	$(1+g)^3\rho^4\sigma^2$	$(1+g)^2\rho^3\sigma^2$	$(1+g)\rho^2\sigma^2$	$\rho\sigma^2$	$1+g$

Each expected cash flow in the table is written as a function of the prior expected cash flows, so that the values in the rows should be read as the coefficients on the expected cash flow in the column headings, for example, $E[C_1]=C_0(1+g)$. Writing the terms out in this fashion allows a simplification in the spirit of Miller and Modigliani (1961). If, instead of reading across the rows, we look at the columns, we can see that each expected cash flow affects the next period's expected cash flow by $(1+g)$, the following period's expected cash flow by $\rho\sigma^2$, the one after that by $(1+g)\rho^2\sigma^2$, the one after that by $(1+g)^2\rho^3\sigma^2$, etc. As is implicit in the autoregressive process, the influence of the first period's growth rate progressively dies off, as shown by the successive powers that the serial correlation coefficient is raised to. However, note that after the first growth rate, the successive terms are a geometric series that change by $(1+g)\rho$. This allows us to simplify by collecting all the terms involving the expected cash flows in each column, that is by going down the columns in the table, instead of by going across each row.

For example, take the first set of coefficients on the C_0 term. The first coefficient is $(1+g)$, the second is $\rho\sigma^2$, the third $(1+g)\rho^2\sigma^2$, etc. Factoring the term $\rho\sigma^2$ from the second term on, we have a simple geometric series growing by $\rho(1+g)$. As a result, we can collapse all of the future terms involving C_0 into one term[3]

$$C_0\left[1+g+\frac{\rho\sigma^2}{(1-(1+g)\rho)}\right] \tag{13}$$

Since all the expected cash flows have the same evolution of terms, they will all be adjusted by this same term. We define the term in the square brackets as one plus the adjusted growth rate g^*. Using this adjusted growth rate, the table can be rewritten as follows,

	C_0	$E[C_1]$	$E[C_2]$	$E[C_3]$	$E[C_4]$
$E[C_1]$	$1+g^*$				
$E[C_2]$		$1+g^*$			
$E[C_3]$			$1+g^*$		
$E[C_4]$				$1+g^*$	
$E[C_5]$					$1+g^*$

This table has equivalent present value as the previous table, but a different meaning. The expected cash flow in time period 1 is now equal to the current cash flow times one plus the

[3] This assumes convergence in the geometric series, if the growth rate is discounted the approximation is marginally worse.

"adjusted" growth rate. However, the adjusted growth rate is not an actual growth rate, since from Eq. (3), $E(C_1) = C_0(1+g)$ and there are *no* adjustments needed for the first period's growth rate. What g^* includes is the impact of the first period's uncertain growth rate on the expected value of all subsequent cash flows. Since this impact is the same for all subsequent expected cash flows, we can collapse its impact into one "average" growth rate that gives approximately the correct present value. In this way, the value of the cash flows is just

$$V = \frac{C_0(1+g^*)}{K-g^*} \tag{14}$$

which is identical in structure to the conventional constant growth model, except for the use of the adjusted growth rate g^* However, the adjusted growth rate is an artifact, at no point will the expected cash flows actually be expected to increase at this rate.

How significant is the adjustment? Well, suppose the current cash flow is a dollar, the expected long-run growth rate 5% and the discount rate 10%. Using the standard constant growth model would indicate a price of \$21. However, if the serial correlation coefficient is 0.4 and the standard deviation of the growth rate 10%, then the adjusted growth rate as 5.69% and the "true" value should be \$24.5. By ignoring the serial correlation in the growth rate, all future expected cash flows are underestimated, causing the estimated value to be lower. Conversely, if the value is \$24.5, using the standard DCF model would estimate the investor's required rate of return at 9.28% or 72 basis points *below* the assumed discount rate of 10%. Regardless of the stock price, application of the "certainty" DCF model will underestimate the discount rate, since it underestimates the expected cash flow stream.

With a 10% discount rate and a 5% long-run expected growth rate, the following table highlights how significant the bias (in basis points) involved in ignoring the serial correlation can be.

Variance	Serial correlation		
	0.2	0.4	0.6
0.0025	6	17	40
0.01	25	69	162
0.04	101	275	648

With low levels of serial correlation and growth rate uncertainty, the underestimation is relatively low, only six basis points for a 0.2-serial correlation coefficient and a standard deviation of the growth rate of 5%. As either the correlation coefficient or the growth rate uncertainty increase, the bias also increases. With a standard deviation of the growth rate of 20% and a correlation coefficient of 0.6, the bias is 648 basis points or an adjusted growth rate more than twice the expected growth rate. It is an empirical question as to where in the previous table reasonable estimates of the bias actually lie.

Another way of looking at the bias in the growth rate is to consider the impact on the valuation multiplier, which is just.

$$\frac{V}{C_0} = \frac{(1+g^*)}{K-g^*} \tag{15}$$

Again, with a 10% discount rate and 5% long-run expected growth rate, the standard multiplier would be 21 × cash flow. However, the multipliers with the adjusted growth rate are

Variance	Serial correlation		
	0.2	0.4	0.6
0.0025	21.3	21.8	22.9
0.01	22.2	24.5	31.6
0.04	26.6	48.1	udf

which range from minor adjustments in the case of limited serial correlation to a situation where the multiplier is undefined (udf) since the adjusted growth rate of 11.48% exceeds the discount rate of 10%. As a result, the geometric series in the constant growth model does not converge. These larger multiplier values indicate that reasonable values for the serial correlation coefficient and the variance on the growth rate process can quickly make reasonable estimates of the discount rate too low to determine a finite price. Since prices obviously exist this in turn implies that the standard application of the constant growth model can seriously underestimate equity costs.

An interesting implication of these results is that there may be nothing wrong with forecasting that corporate profits and cash flows can grow at a rate greater than the expected growth rate of the economy in perpetuity. In turn, it implies that the constraint imposed by aggregate profits, etc., in relation to GDP is not as obvious as it appears to be. This is because much of the variation in firm level profits consists of a zero sum game, that nets out at the aggregate level. This in turn may render much of the discussion of the size of the equity risk premium derived from models like the constant growth model, applied to aggregate values for the S&P500, that do not take into account parameter uncertainty, of questionable value.

4. Simulation of the model

The general model is very easy to simulate and has the added advantage of showing how the estimation bias changes on a year by year basis. Consistent with the earlier discussion, a long-run growth rate of 5% and a discount rate of 10% is assumed with an initial cash flow of $1. The constant growth model then implies a value of $21. Quattro Pro was used to simulate 100 "years" of annual cash flows 1000 times from a normal distribution with an exact mean of zero and an exact standard deviation of 10% for each year. The same random numbers were used in all the simulations to preserve consistency. Since only 100 years of cash flows were generated, the theoretical and simulated value of the expected cash flows without any serial correlation were both $20.80, the difference of $0.20 representing the present value of cash flows from year 101 to infinity.

Our interest focuses on the bias in estimating the expected cash flow. What is of interest, for example, is the covariance term in Eq. (9). However, this will change with the mean of the expected cash flow. For this reason, I rewrite Eq. (9) as follows

$$E[C_3] = E[C_2]\left(1 + g + \mathrm{Cov}\left(\frac{C_2}{E[C_2]}, g_3\right)\right) \tag{16}$$

the estimation bias for each period is then the covariance of the standardized prior cash flow with the contemporaneous growth rate where the standardization is achieved by dividing by the expected cash flow. In the simulation for each year, 1000 values are calculated for the growth rate which is then used to calculate 1000 values for the cash flow. The covariance in Eq. (16) can then be calculated each year from the 1000 simulated variables for the growth rate and the prior period's standardized cash flow.

For the base case with no serial correlation, Fig. 1 indicates the covariance expressed in terms of basis points. Without serial correlation, there is no bias in using the expected growth rate to compound the future cash flows and the graph shows this. The average of the covariance terms is zero. However, since the uncertainty in the cash flow and the growth rate both increase across time, the estimated value of the covariance becomes increasingly more volatile around the mean of zero. The graph's value is simply to show the random sampling error introduced by the particular 100,000 random numbers used.

Fig. 2 indicates the covariance term using a serial correlation coefficient of 0.4 and a standard deviation of the growth rate of 0.1. Unlike the earlier figure, there is no fluctuation around zero. Instead, the covariance builds rapidly to around 0.5, and then becomes increasingly more volatile However, the mean continues to differ significantly from zero.

The "correct" approach is to calculate each period's true expected cash flows, using the covariance term in Eq. (13), graphed above. This, however, necessitates varying the growth rate period by period, even though the expected growth rate is a constant 5%. The adjusted growth rate from Eq. (13) is simply an approximation for the present value of

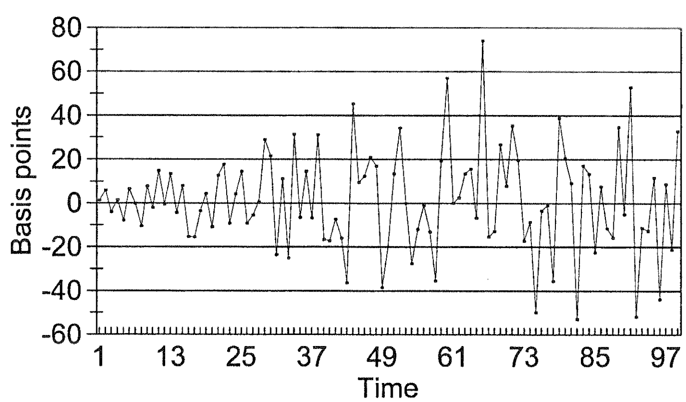

Fig. 1. Covariance: growth with lagged cash flow.

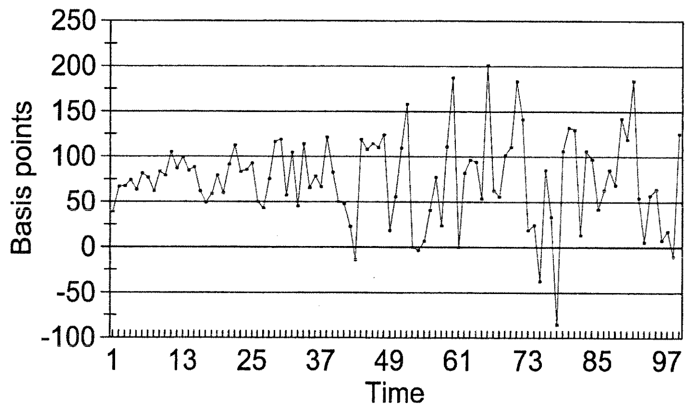

Fig. 2. Covariance: growth with lagged cash flow.

these deviations from the expected growth rate. For our example, the adjusted growth rate is an extra 69 basis points, which can then be inserted into the constant growth model. If this is done the value of the first 100 years of cash flows is \$24.1 versus the simulation value of \$24.3 and the constant "certainty" growth value of \$20.80. The simulation, thus, supports the prior development of the adjusted growth rate to indicate that failing to account for the uncertainty in the future growth rate in the presence of serial correlation undervalues the expected future cash flows, with our simulation this undervaluation is about 10%.

The adjusted growth rate approximation worsens as the serial correlation coefficient increases. The following table indicates that with a serial correlation coefficient of 0.1, both the simulation value and that obtained by using the adjusted growth rate are \$21.3, as compared to the certainty estimate of \$20.80. However, for a serial correlation coefficient of 0.7, the simulation estimate of value of \$63.8 is half as much again as that estimated using the adjusted growth rate and three times that of the certainty estimate, which remains at \$20.8. The reason for the growing discrepancy is that as the serial correlation coefficient increases, so too does the volatility of the growth rate, which causes the distribution of the cash flow to become increasingly nonnormal. This makes the Steven's approximation less useful.

Serial correlation	Simulation	Adjusted growth rate
0	20.8	20.8
0.1	21.3	21.3
0.2	21.9	21.9
0.3	22.8	22.8
0.4	24.3	23.5
0.5	27.1	26.2
0.6	34.1	30.2
0.7	63.8	40.4

So far, the analysis has focussed on the valuation of the expected future cash flows using an externally derived discount rate. However, in practice the constant growth model, in particular, is used as much for estimating discount rates as it is for valuation. Incorporating parameter uncertainty then simply involves reversing the valuation equation and solving for the discount rate, where the cash flows are just the dividends received by the investor,

$$K = \frac{d_0\left(1 + g + \frac{\rho\sigma^2}{(1-(1+g)\rho)}\right)}{P_0} + \frac{\rho\sigma^2}{(1-(1+g)\rho} \tag{17}$$

Suppose the dividend is a dollar and the stock price \$35 and the expected growth rate 5%. The certainty DCF model would estimate the investor's discount rate as a 3% expected dividend yield and a 5% growth rate or 8%. However, the expected stream of dividends is underestimated since the effects of serial correlation are not taken into account. In this case, the discount rate is higher, since a higher rate is needed to discount the higher stream of expected cash flows back to the same \$35 price.

If we use the same parameter values as before, the correct discount rates are

Variance	Serial Correlation		
	0.2	0.4	0.6
0.0025	8.06	8.18	8.42
0.01	8.26	8.71	9.67
0.04	9.04	10.83	14.67

These estimates largely just capture the increased growth rate estimates. However, they indicate that the bias in estimating the equity cost for low serial correlation coefficient firms is relatively small. However, as before as either the serial correlation coefficient or the variance in the growth rate increases, there is a significant increase in the true equity cost estimate. For the risky firm with a 20% standard deviation and a serial correlation coefficient of 0.6, the equity cost is 83% higher at 14.67%.

5. Empirical estimates

Serial correlation is a fact of life in most economic series. For example, using data on the earnings and dividends per share from the Toronto Stock Exchange 300 index gave the following estimates for Eq. (7)

	Long-run g	ρ	R^2 (%)	Standard deviation	
				σ_g^2	σ_ϵ^2
Dividends	0.049	0.51	24.4	0.086	0.0714
Earnings	0.006	0.54	22.3	0.262	0.235

The standard deviation of the error term for the aggregate dividend process is 7.14%, which causes an overall standard deviation for the dividend growth rate of 8.6%. The dividend growth rate in aggregate has a serial correlation coefficient of 0.51 and the estimated long-run dividend growth rate is 4.9%. If the aggregate dividends are forecast using a 5.0% long-run expected growth rate, it would significantly underestimate the actual growth rate in the expected dividend stream which by Eq. (13) has an adjusted growth rate 56 basis points higher at 5.56%.

The standard deviation of the error term for the aggregate earnings per share growth rate is 23.5% or 26.2% for the overall growth rate. This 23.5% standard deviation is considerably larger than the 10% assumed in the simulation, while that for the dividend process is slightly smaller. However, the serial correlation in the earnings growth rate is almost the same at 0.54. Interestingly, the estimate for the long-run earnings growth rate is only 0.6%, reflecting the ending date of 1992, which was in the depths of the recession as Canadian companies went through a wrenching adjustment to free trade with the US during a normal cyclical downturn.[4] Inserting the serial correlation coefficient and standard deviation into a long-run 5% growth rate increases the growth rate in expected earnings by 6.9–11.9%.

Looking at the aggregate empirical data, what is clear is the greater instability of aggregate earnings than dividends. This is due to the tendency of corporations to smooth their dividends as first noted by Lintner (1956). With the same serial correlation coefficient, the earnings per share are three times as volatile, meaning that the growth rate in earnings per share, all else constant, will be three times as large.[5] In estimating growth rates, analysts frequently use earnings, as well as dividend growth rates, as proxies for the long-run growth rate. However, what the simulation and empirical data show is that expected dividends and earnings per share can grow at different rates due to the greater instability of earnings than dividends. Miller and Modigliani (1961) showed that valuing a firm by discounting dividends, earnings and cash flow should all give the same value as long as earnings and cash flow are correctly defined. However, what the above points out is that to achieve this, in the face of Lintner style dividend smoothing, may require the use of different discount rates.

The empirical data and simulation also show that for two firms with the same long-run growth rate the "risky" firms' expected dividends/earnings/cash flow will grow at a greater rate than a less risky one, all else constant. Moreover, this rate can significantly exceed the long-run growth rate in the economy. As a result, the familiar constraint that the firm can not grow in perpetuity at say 8%, while the economy grows at 5%, since it eventually will become the economy, may be wrong.[6] Quite the reverse, risky firms expected dividends/earnings and cash flow, etc., must be expected to grow significantly more than the expected growth rate of the economy in perpetuity. When these growth rates

[4] The 1992–1994 recession was much more severe in Canada than the US due to the free trade adjustment. During this period, *aggregate* corporate profits were close to zero.

[5] If earnings grow by either +100% or −50% and dividends by either +50% or −33.3%, the expected arithmetic growth rate is +25% for earnings and +8.33% for dividends, but the compound growth rate (geometric) is zero for both. The different arithmetic growth rates have the same geometric growth rates, so that over time earnings and dividends can expect to grow at different rates.

[6] This constraint better applies to the compound growth rate, not the arithmetic or per period long run growth rate or the growth rate in expected dividends or earnings. However, there is still an implicit constraint that arises as a result of a series of positive growth rates.

are serially correlated, the dividends and earnings can be expected to grow by several orders of magnitude greater than the economy's growth rate and as a result have to be discounted at much higher rates to achieve the same market value.

6. Conclusions

Section 4 showed that assuming serially correlated growth rates can introduce significant changes in the forecast expected cash flows. As a result, using the DCF model without taking this into account will significantly underestimate values and discount rates. This applies whether the cash flows that are being estimated are from a constant growth or any other multistage growth model, since what is important is not the expected growth rate, but the serial correlation in the growth rate. Further, these comments apply to any discounting model, whether it is based on dividends, earnings or cash flow and whether it is an investor based or corporate based model.

For a variety of reasons, firms smooth their dividends, generally by using short term borrowing. In this case, the uncertainty and serial correlation in a firm's future dividends is generally less than in a firm's earnings, which again are less than in their free cash flows. Given this observation, there is no reason to believe that parameter uncertainty is the same for dividends, earnings and cash flow, in which case even though value may be the same there is no reason to believe that the same discount rate should be used to discount all three. How the discount rate should vary when discounting dividends, earnings and cash flow will depend on the firm's empirical dividend policy. Investigating the implications of dividend policy on parameter uncertainty is a topic of current research.

Generally, in valuation, we ignore parameter uncertainty and just forecast expected values based on current values and some forecast expected growth rate. This paper has shown that serious errors arise with this procedure, since with serial correlation the expected cash flows can be significantly higher than the estimates resulting from simple extrapolation at the expected growth rate. This in turn implies that valuations and equity costs are biased low without making the adjustments for serial correlation developed in this paper. Finally, it explains why two firms with the same dividend yield and expected growth rate can have different equity costs, and why in general there may be no simple relation between the perpetual long-run growth rate in a firm's expected cash flows and the expected growth rate in the economy.

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References

Fama, E., 1970. Multi-period consumption—investment decisions. American Economic Review 60, 163–174.
Gordon, M., 1962. The Investment, Financing and Valuation of the Corporation. Richard D. Irwin, Homewood, Illinois.

Lintner, J., 1956. The distribution of incomes of corporations among dividends, retained earnings and taxes. American Economic Review 28, 97–113.
Miller, M., Modigliani, F., 1961. Dividend policy, growth and the valuation of shares. Journal of Business 34, 411–433.
Myers, S., Borucki, L., 1994. Discounted cash flow estimates of the cost of equity capital. Financial Markets and Institutions. Blackwell Publishing, Oxford.
Myers, S., Turnbull, S., 1977. Capital budgeting and the capital asset pricing model: good news and bad news. Journal of Finance 32, 321–332.
Mossin, J., 1968. Optimal multi-period portfolio choices. Journal of Business 41, 215–229.
Stevens, G., 1971. Two problems in portfolio analysis: conditional and multiplicative random variables. Journal of Financial and Quantitative Analysis 6-5, 1235–1250.